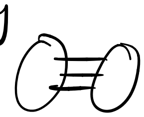
Recap

l_0-sampling : linear sketch

graph connectivity

"semi-streaming"

K- Connectivity

 $|E(S,\bar{S})| \geq K \quad \forall \; S \subset V$

1- connectivity ≡ Connectivity

Non-sketch algorithm:

① For $i = 1$ to K, let F_i be spanning forest of $\left(V, E \setminus \bigcup_{j=1}^{i-1} F_j\right)$

② Then $(V, F_1 \cup \dots \cup F_K)$ is K-edge connected iff $G(V,E)$ is K-edge connected

Correctness: For any cut,

Either: every F_i contains an edge across this cut

OR $F_1 \cup \dots F_{i-1}$ already contain **all** edges across this cut

Hence: if $F_1 \cup \dots \cup F_K$ does not contain all edges across the cut, it contains at least K of them.

Such a set of edges ≜ K-skeleton

Emulation via sketches:

Compute sketches in streaming fashion

then emulate algorithm on compressed rep^n

If we have sketch $\mathcal{A}(G)$, need $\mathcal{A}(G-F)$

F: identified edges

$$\mathcal{A}(G-F) = \mathcal{A}(G) - \mathcal{A}(F)$$

① In one pass, compute $\mathcal{A}^1(G) \ldots \mathcal{A}^k(G)$
k indpt sketches for spanning forest

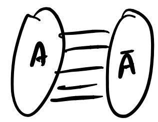
② Post-processing: emulate orignal algo.
For $i \in [k]$ construct spanning forest F_i of
$(V, E \setminus F_1 \ldots \cup F_{i-1})$ using
$$\mathcal{A}^i(G - F_1 - F_2 \ldots - F_{i-1}) = \mathcal{A}^i(G) - \sum_{j=1}^{i-1} \mathcal{A}_i(F_j)$$

Space: for each spanning forest sketch: $O(n \text{ polylog } n)$
k-connectivity: $(kn \text{ polylog } n)$

Min-Cut exactly upto k
larger values of min-cut ?
Introduce some machinery

Graph Sparsification:
sparse repⁿ of graph that allows estimation of connectivity properties of original graph

[Benczür, Karger] Cut-sparsifier
Weighted subgraph H is a $(1 \pm \varepsilon)$-cut sparsifier of graph G if
$$\lambda_A(H) = (1 \pm \varepsilon)\lambda_A(G) \quad \forall A \subset V$$
$\lambda_A(G)$ and $\lambda_A(H)$ is wt. of cut $(A, \bar{A})$ in G and H



$$|E_G(A, \bar{A})| = \lambda_A(G)$$
$$w_H(A, \bar{A}) = \lambda_A(H)$$

[Spielman, Teng]: Spectral sparsification based on approximation of Laplacian of graph

Laplacian: Laplacian of undirected weighted graph $H(V, E, w)$ is matrix $L_H \in \mathbb{R}^{n \times n}$

$$L_H(i,j) = \begin{cases} -w(i,j) & i \neq j \\ \sum_{(i,k) \in E} w(i,k) & \text{if } i = j \end{cases}$$

$w(i,j)$: wt. of edge (i,j)
$= 0$ if no such edge

Spectral Sparsifier: Weighted subgraph H is a $(1+\varepsilon)$-spectral sparsifier of G if $x^T L_H x = (1 \pm \varepsilon)\, x^T L_G x \quad \forall x \in \mathbb{R}^n$

L_G, L_H: Laplacians of G & H

Observation $\quad x^T L_G x = \sum_{(i,j) \in E} w_{ij} (x_i - x_j)^2$

Graph Laplacian is sum of edge Laplacians

$$L_G = \sum_{(i,j) \in E} L_{(i,j)} = \sum_{(i,j) \in E} w_{ij} (e_i - e_j)(e_i - e_j)^T \qquad \begin{array}{c|cc} & i & j \\ \hline i & w_{ij} & -w_{ij} \\ j & -w_{ij} & w_{ij} \end{array}$$

$$x^T L_G x = \sum_{(i,j) \in E} w_{ij} (x_i - x_j)^2 \qquad x^T \qquad x$$

When $x \in \{0,1\}^n \quad x^T L_G x = w(E_G(S, \bar{S}))$
$S = \{i : x_i = 1\}$

Sparsification via sampling:

① Sample each edge with probability p_e

② Weight each sampled edge $\frac{1}{p_e}$

Size of cut preserved in expectation

If p_e large enough $\Rightarrow$ high probability bounds

[Karger] $p_e \geq q := \min\left\{1, \frac{c_1 \log n}{\lambda \cdot \varepsilon^2}\right\}$

λ : size of min-cut

then resulting graph is cut sparsifier w.h.p.

[Spielman, Srivastava] $p_e \geq \min\left\{1, \frac{c_2 \, r_e \log n}{\varepsilon^2}\right\}$

then resulting graph is a spectral sparsifier w.h.p.

r_e : effective resistance of e

view graph as electrical circuit $V = IR$

r_{uv} = voltage diff between u & v needed for one amp current to flow from u & v

Back to min cut :

Karger sampling + k-skeleton construction

G_i : graph sampled from G by including each edge w. prob $\frac{1}{2^i}$

H_i = skeleton$_k(G)$ $\quad k = \frac{3c_1 \log n}{\varepsilon^2}$

<u>Claim :</u> $j = \min\{i : \text{mincut}(H_i) < k\}$

$2^j \, \text{mincut}(H_j) = (1 \pm \varepsilon)\lambda$

<u>Proof</u> : Recall Karger's condⁿ

$$\frac{1}{2^i} = p_e \geq \frac{c_1 \log n}{\lambda \varepsilon^2} = q$$

$$i \leq \left\lfloor \log_2 \frac{1}{q} \right\rfloor$$

$$2^i \cdot \text{mincut}(G_i) = (1 \pm \varepsilon) \, \text{mincut}(G)$$

For $i = \left\lfloor \log_2 \frac{1}{q} \right\rfloor$

If $\text{mincut}(G)$ achieved for $(A, \bar{A})$

$$E[|E_{G_i}(A,\bar{A})|] = \frac{\lambda}{2^i} \leq 2q\lambda = \frac{2c_1 \log n}{\varepsilon^2}$$

w.h.p. $|E_{G_i}(A,\bar{A})| < \frac{3c_1 \log n}{\varepsilon^2} = K$

w.h.p. $\text{mincut}(G_i) < K$

Spectral Sparsifiers in streaming model.

Simplification: edge arrivals only

[Spielman, Srivastava] $(1+\varepsilon)$-spectral sparsifier with $O(\frac{n \log n}{\varepsilon^2})$ edges via sampling

[Batson, Spielman, Srivastava]

Deterministic algorithm to get $(1+\varepsilon)$-spectral sparsifier with $O(\frac{n}{\varepsilon^2})$ edges

"Merge & Reduce" Approach

$(1+\gamma)$-spectral sparsifier construction as black box

$\text{size}(\gamma) = O(\frac{n}{\gamma^2})$ # edges

2 properties:

<u>Mergeable</u>: H_1, H_2 α-spectral sparsifiers of G_1, G_2 then $H_1 \cup H_2$ is α-spectral sparsifier of $G_1 \cup G_2$

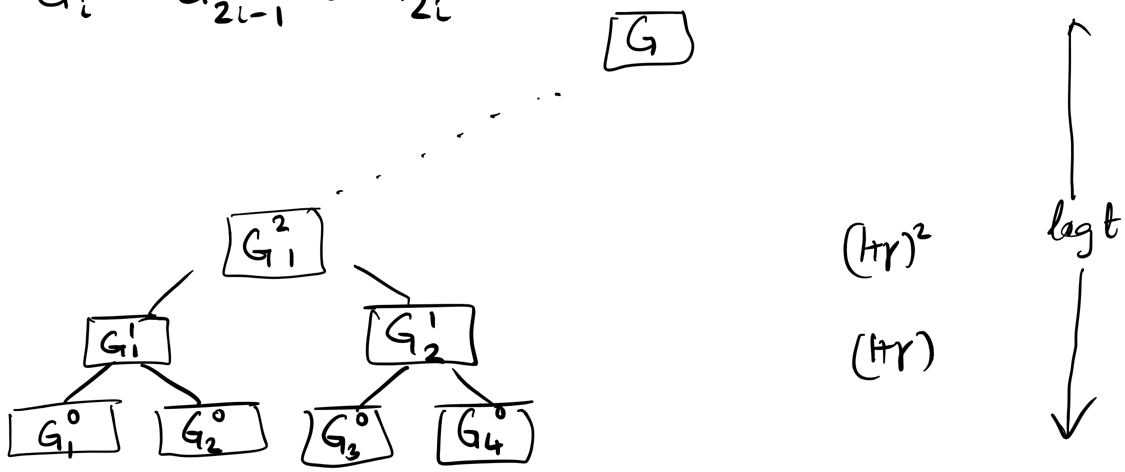
Composable: if H_3 is α-spectral sparsifier for H_2, H_2 is β-spectral sparsifier for H_1 then H_3 is $\alpha\beta$ spectral sparsifier for H_1

Hierarchical partition of stream of m edges

partition stream into $t = \frac{m}{\text{size}(\gamma)}$ segments of $\text{size}(\gamma)$ edges

G_i^0 : graph corresponding to i^{th} segment of edges

$G_i^j = G_{2i-1}^{j-1} \cup G_{2i}^{j-1}$



For each G_i^j : weighted subgraph H_i^j using sparsification algo $\mathcal{A}$

$H_i^0 = G_i^0 \qquad H_i^j = \mathcal{A}\left(H_{2i-1}^{j-1} \cup H_{2i}^{j-1}\right)$

H_i^j is $(1+\gamma)^j$ spectral sparsifier of G_i^j

$\Rightarrow (1+\gamma)^{\log_2 t}$ sparsifier of G

$\gamma = \frac{\varepsilon}{2\log_2 t} \Rightarrow (1+\varepsilon)$ - spectral sparsifier

At most $O(\text{size}(\gamma) \cdot \log_2 t) = O\left(\frac{n \log^3 n}{\varepsilon^2}\right)$ edges at any time